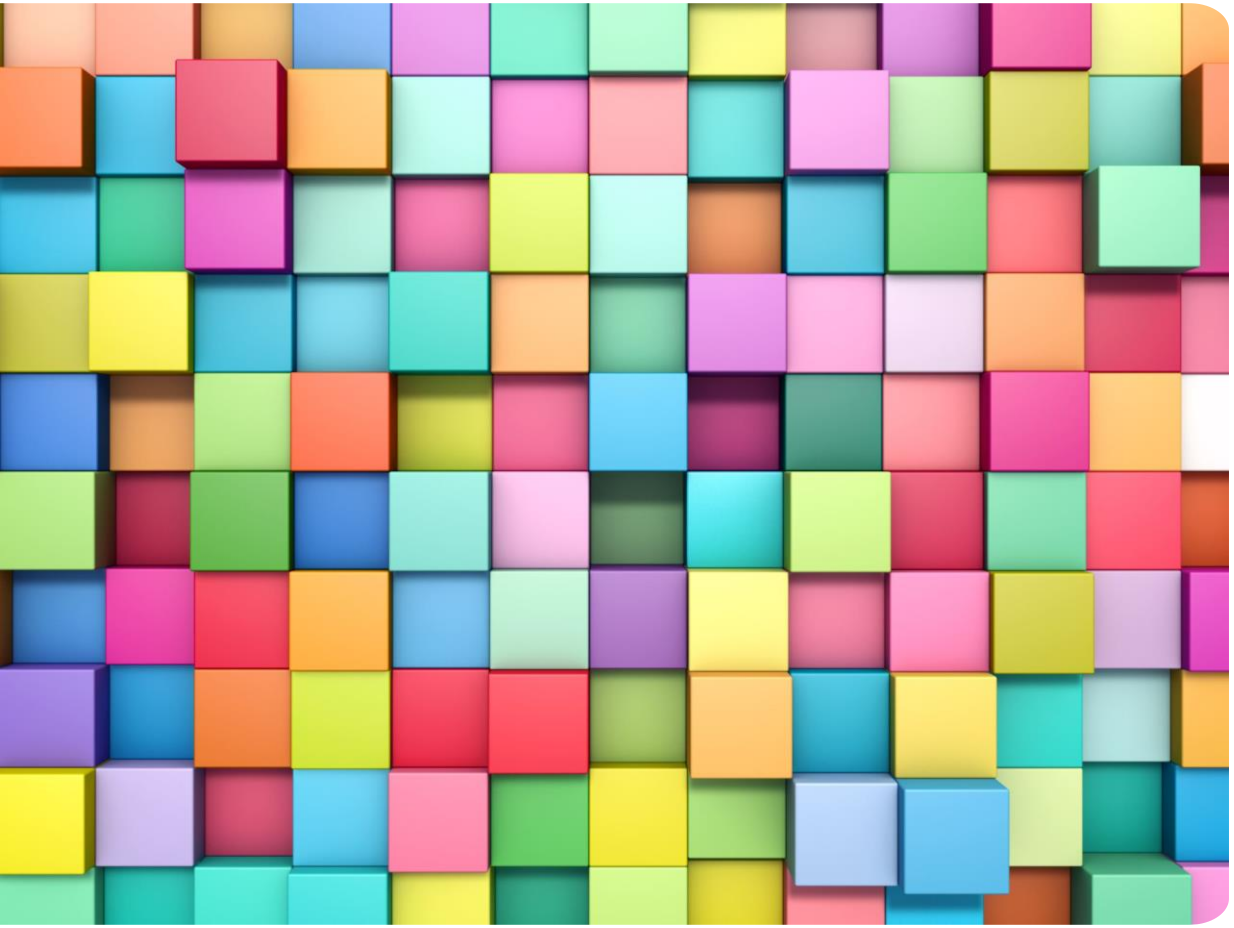

NYQUIST CRITERION and Bode diagrams

Digital control

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The Nyquist criterion

The Nyquist criterion is one of the widely used stability analysis techniques in the s-plane, based on the frequency response of the system. To determine the frequency response of a continuous system transfer function $G(s)$, we replace s by $j\omega$ and use the transfer function $G(j\omega)$. In the s-plane, the Nyquist criterion is based on the plot of the magnitude $|GH(j\omega)|$ against the angle $\angle GH(j\omega)$ as ω is varied.

In a similar manner, the frequency response of a transfer function $G(z)$ in the z-plane can be obtained by making the substitution $z = e^{j\omega T}$. The Nyquist plot in the z-plane can then be obtained by plotting the magnitude of $|GH(z)|_{z=e^{j\omega T}}$ against the angle $\angle GH(z)|_{z=e^{j\omega T}}$ as ω is varied. The criterion is then

$$Z = N + P,$$

The Nyquist criterion

where N is the number of clockwise circles around the point -1, P the number of poles of $GH(z)$ that are outside the unit circle, and Z the number of zeros of $GH(z)$ that are outside the unit circle.

For a stable system, Z must be equal to zero, and hence the number of anticlockwise circles around the point -1 must be equal to the number of poles of $GH(z)$.

If $GH(z)$ has no poles outside the unit circle then the criterion becomes simple and for stability the Nyquist plot must not encircle the point -1.

The Nyquist criterion: Example

Example 8.11

The transfer function of a closed-loop sampled data system is given by

$$\frac{G(z)}{1 + GH(z)},$$

where

$$GH(z) = \frac{0.4}{(z - 0.5)(z - 0.2)}.$$

Determine the stability of this system using the Nyquist criterion. Assume that $T = 1$ s.

The Nyquist criterion: Example

Solution

Setting $z = e^{j\omega T} = \cos \omega T + j \sin \omega T = \cos \omega + j \sin \omega$,

$$G(z)|_{z=e^{j\omega T}} = \frac{0.4}{(\cos \omega + j \sin \omega - 0.5)(\cos \omega + j \sin \omega - 0.2)}$$

or

$$G(z)|_{z=e^{j\omega T}} = \frac{0.4}{(\cos^2 \omega - \sin^2 \omega - 0.7 \cos \omega + 0.1) + j(2 \sin \omega \cos \omega - 0.7 \sin \omega)}.$$

This has magnitude

$$|G(z)| = \frac{0.4}{\sqrt{(\cos^2 \omega - \sin^2 \omega - 0.7 \cos \omega + 0.1)^2 + (2 \sin \omega \cos \omega - 0.7 \sin \omega)^2}}$$

and phase

$$\angle G(z) = \tan^{-1} \frac{2 \sin \omega \cos \omega - 0.7 \sin \omega}{\cos^2 \omega - \sin^2 \omega - 0.7 \cos \omega + 0.1}.$$

Table 8.2 lists the variation of the magnitude of $G(z)$ with the phase angle. The Nyquist plot for this example is shown in Figure 8.10. Since $N = 0$ and $P = 0$, the closed-loop system has no poles outside the unit circle in the x-plane and the system is stable.

The Nyquist diagram can also be plotted by transforming the system into the w-plane and then using the standard s-plane Nyquist criterion. An example is given below.

The Nyquist criterion: Example

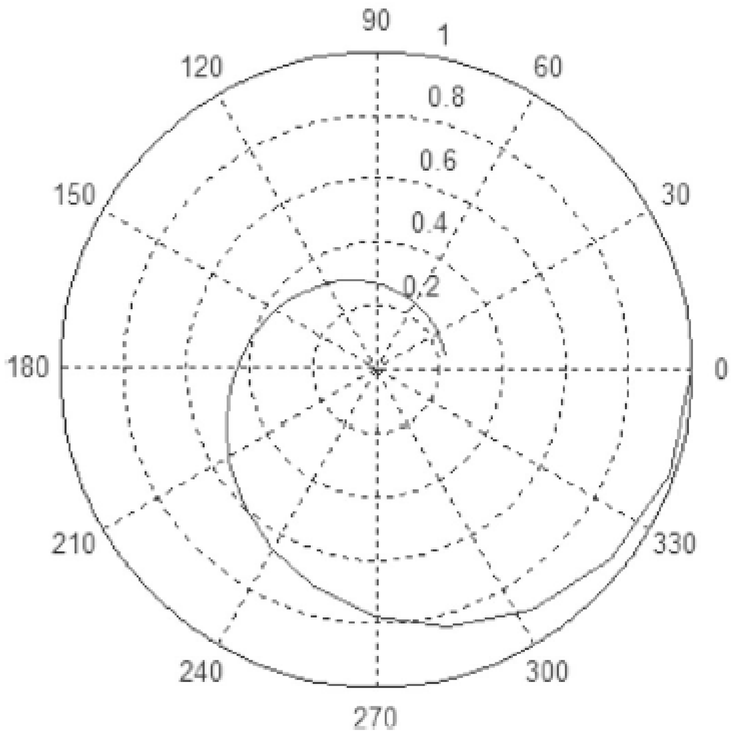


Figure 8.10 Nyquist plot for Example 8.11

Table 8.2 Magnitude and phase of $G(z)$

w	$\|G(z)\|$	$\angle G(z)$
0	1.0000E+000	0
1.0472E−001	9.8753E−001	−1.9430E+001
2.0944E−001	9.5248E−001	−3.8460E+001
3.1416E−001	9.0081E−001	−5.6779E+001
4.1888E−001	8.3961E−001	−7.4209E+001
5.2360E−001	7.7510E−001	−9.0690E+001
6.2832E−001	7.1166E−001	−1.0625E+002
7.3304E−001	6.5193E−001	−1.2096E+002
8.3776E−001	5.9719E−001	−1.3492E+002
9.4248E−001	5.4789E−001	−1.4820E+002
1.0472E+000	5.0395E−001	−1.6089E+002
1.1519E+000	4.6505E−001	−1.7308E+002
1.2566E+000	4.3075E−001	1.7518E+002
1.3614E+000	4.0058E−001	1.6384E+002
1.4661E+000	3.7408E−001	1.5283E+002
1.5708E+000	3.5082E−001	1.4213E+002
1.6755E+000	3.3044E−001	1.3168E+002
1.7802E+000	3.1259E−001	1.2147E+002
1.8850E+000	2.9698E−001	1.1146E+002
1.9897E+000	2.8337E−001	1.0162E+002
2.0944E+000	2.7154E−001	9.1945E+001
2.1991E+000	2.6130E−001	8.2401E+001
2.3038E+000	2.5250E−001	7.2974E+001
2.4086E+000	2.4501E−001	6.3646E+001
2.5133E+000	2.3872E−001	5.4404E+001
2.6180E+000	2.3354E−001	4.5232E+001
2.7227E+000	2.2939E−001	3.6118E+001
2.8274E+000	2.2622E−001	2.7050E+001
2.9322E+000	2.2399E−001	1.8015E+001
3.0369E+000	2.2266E−001	9.0018E+000

The Nyquist criterion: Example

The Nyquist diagram can also be plotted by transforming the system into the w-plane and then using the standard s-plane Nyquist criterion. An example is given below.

Example 8.12

The open-loop transfer function of a unity feedback sampled data system is given by

$$G(z) = \frac{z}{(z-1)(z-0.4)}.$$

Derive expressions for the magnitude and the phase of $|G(z)|$ by transforming the system into the w-plane.

Solution

The w transformation is defined as

$$z = \frac{1+w}{1-w}$$

which gives

$$G(w) = \frac{(1+w)/(1-w)}{((1+w/1-w)-1)((1+w/1-w)-0.4)} = \frac{1+w}{2w(0.6+1.4w)},$$

or

$$G(w) = \frac{1+w}{1.2w+2.8w^2}.$$

The Nyquist criterion: Example

But since the w-plane can be regarded an analogue of the s-plane, i.e. $s = \sigma + jw$ in the s-plane and $w = \sigma_w + jw_w$ in the z-plane, for the frequency response we can set

$$w = jw_w$$

which yields

$$G(jw_w) = \frac{1 + w_w}{1.2jw_w - 2.8w^2{}_w}.$$

The magnitude and the phase are then given by

$$|G(jw_w)| = \frac{1 + w_w}{\sqrt{(1.2w_w)^2 + \left(2.8w^2{}_w\right)^2}}$$

and

$$\angle G(jw_w) = \tan^{-1}\frac{1.2}{2.8w_w},$$

where w_w is related to w by the expression

$$w_w = \tan\left(\frac{wT}{2}\right).$$

Bode diagram

The Bode diagrams used in the analysis of continuous-time systems are not very practical when used directly in the *z*-plane.
This is because of the $e^{j\omega T}$ term present in the sampled data system transfer functions when the frequency response is to be obtained.

Gain Margin
The greater the **Gain Margin** (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.

We can usually read the gain margin directly from the Bode plot (as shown in the diagram above). This is done by calculating the vertical distance between the magnitude curve (on the Bode magnitude plot) and the x-axis at the frequency where the Bode phase plot = 180°. This point is known as the **phase crossover frequency**.

The **formula for Gain Margin** (GM) can be expressed as:

$$GM = 0 - G \ dB$$

Where G is the gain. This is the magnitude (in dB) as read from the vertical axis of the magnitude plot at the phase crossover frequency.

https://www.electrical4u.com/bode-plot-gain-margin-ph

Bode diagram

The greater the **Phase Margin** (PM), the greater will be the stability of the system. The phase margin refers to the amount of phase, which can be increased or decreased without making the system unstable. It is usually expressed as a phase in degrees.

We can usually read the phase margin directly from the Bode plot (as shown in the diagram above). This is done by calculating the vertical distance between the phase curve (on the Bode phase plot) and the x-axis at the frequency where the Bode magnitude plot = 0 dB. This point is known as the **gain crossover frequency**.

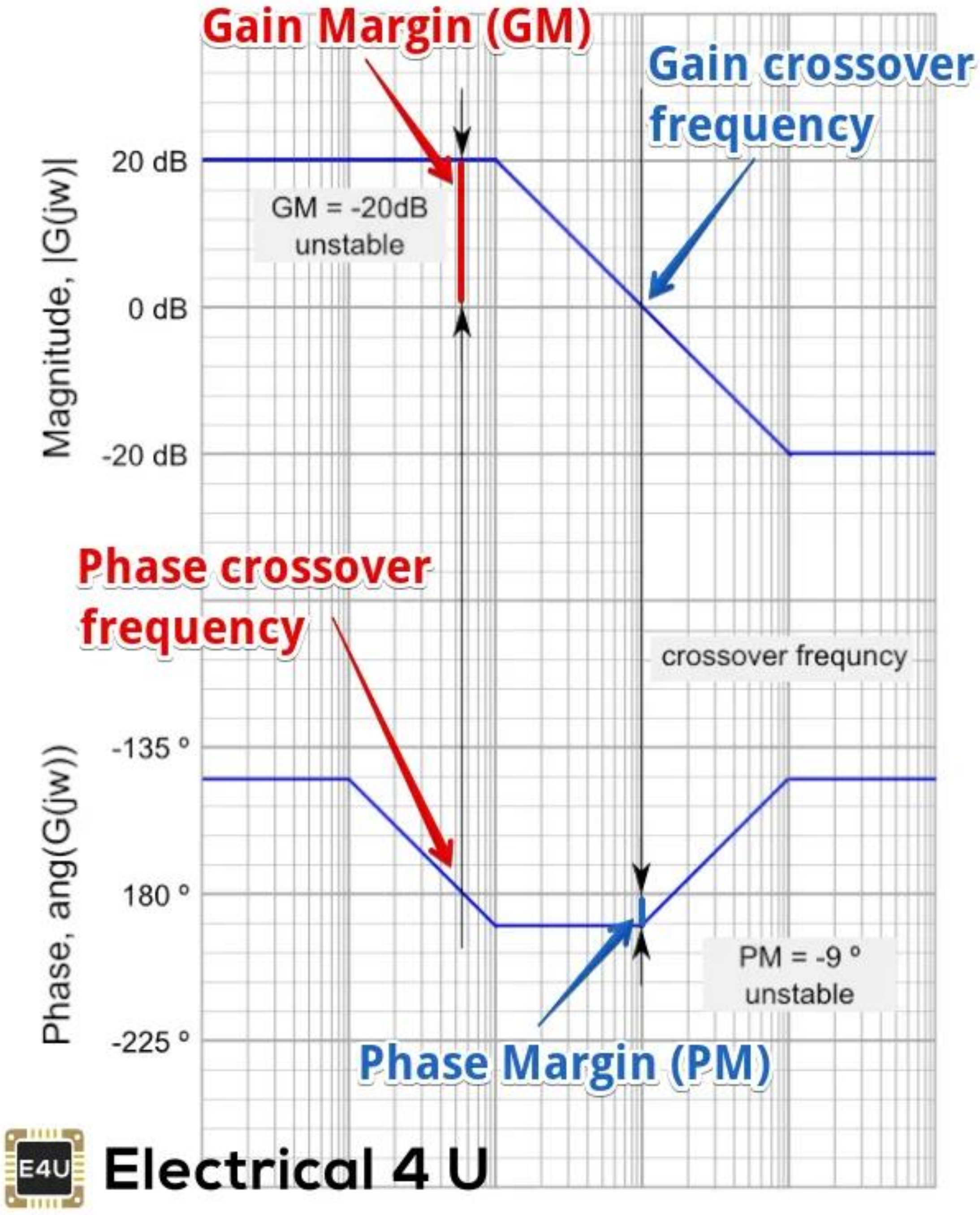
The **formula for Phase Margin** (PM) can be expressed as:

$$PM = \phi - (-180^\circ)$$

Where ϕ is the phase lag (a number less than 0). This is the phase as read from the vertical axis of the phase plot at the gain crossover frequency.

https://www.electrical4u.com/bode-plot-gain-margin-phase-

Bode diagram



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Bode diagram

- **Bode Stability Criterion**
- Stability conditions are given below:

1. For a Stable System: Both the margins should be positive or phase margin should be greater than the gain margin.
2. For Marginal Stable System: Both the margins should be zero or phase margin should be equal to the gain margin.
3. For Unstable System: If any of them is negative or **phase margin** should be less than the gain margin.

Bode diagrams

- It is possible to draw the Bode diagrams of sampled data systems by transforming the system into the *w*-plane by making the substitution:

$$z = \frac{1+w}{1-w},$$

where the frequency in the w-plane (w_w) is related to the frequency in the s-plane (w) by the expression

$$w_w = \tan\left(\frac{wT}{2}\right). \tag{8.8}$$

Bode diagrams

It is common in practice to use a similar transformation to the one given above, known as the w'-plane transformation, which gives a closer analogy between the frequency in the s-plane and the w'-*plane*. The w'-*plane* transformation defined as

$$w' = \frac{2}{T}\frac{z-1}{z+1}, \tag{8.9}$$

or

$$z = \frac{1+(T/2)w'}{1-(T/2)w'}, \tag{8.10}$$

and the frequencies in the two planes are related by the expression

$$w' = \frac{2}{T}\tan\frac{wT}{2}. \tag{8.11}$$

Bode diagrams

Note that for small values of the real frequency (s-plane frequency) such that wT is small, (8.11) reduces to

$$w' = \frac{2}{T}\tan\frac{wT}{2} \approx \frac{2}{T}\left(\frac{wT}{2}\right) = w \tag{8.12}$$

Thus, the w'-plane frequency is approximately equal to the s-plane frequency. This approximation is only valid for small values of wT such that $\tan(wT/2) \approx wT$, i.e.

$$\frac{wT}{2} \leq \frac{\pi}{10},$$

which can also be written as

$$w \leq \frac{2\pi}{10T}$$

or

$$w \leq \frac{w_s}{10}, \tag{8.13}$$

where w_s is the sampling frequency in radians per second. The interpretation of this is that the w'-plane and the s-plane frequencies will be approximately equal when the frequency is less than one-tenth of the sampling frequency.

Bode diagrams

Example 8.13

Consider the closed-loop sampled data system given in Figure 8.11. Draw the Bode diagram and determine the stability of this system. Assume that $T = 0.1$ s.

Solution

From Figure 8.11,

$$G(z) = Z\left\{\frac{1-e^{-sT}}{s}\frac{5}{s+5}\right\} = \frac{1-e^{-0.5}}{z-e^{-0.5}},$$

or

$$G(z) = \frac{0.393}{z-0.606}.$$

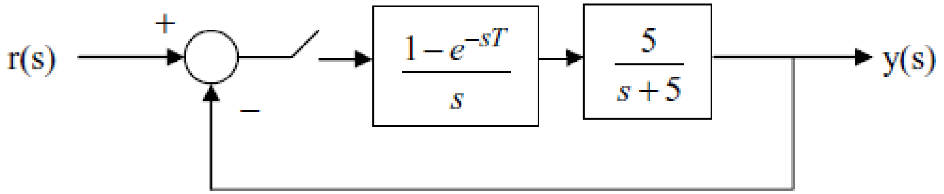

Figure 8.11 Closed-loop system

Bode diagrams

Transforming the system into the w'-plane gives

$$G(w') = \frac{0.393 - 0.0196w}{0.08w - 0.393},$$

or

$$G(w') = \frac{4.9(1 - 0.05w')}{w' + 4.9}.$$

The magnitude of the frequency response and the phase can now be calculated if we set $w' = jv$ where v is the analogue of true frequency ω. Thus,

$$G(jv) = \frac{4.9(1 - 0.05jv)}{jv + 4.9}.$$

The magnitude is

$$|G(jv)| = \frac{4.9\sqrt{1 + (0.25v)^2}}{\sqrt{v^2 + 4.9^2}}$$

and

$$\angle G(jv) = -\tan^{-1}(0.05) - \tan^{-1}\left(\frac{v}{4.9}\right).$$

Bode diagrams

The Bode diagram of the system is shown in Figure 8.12. The system is stable.

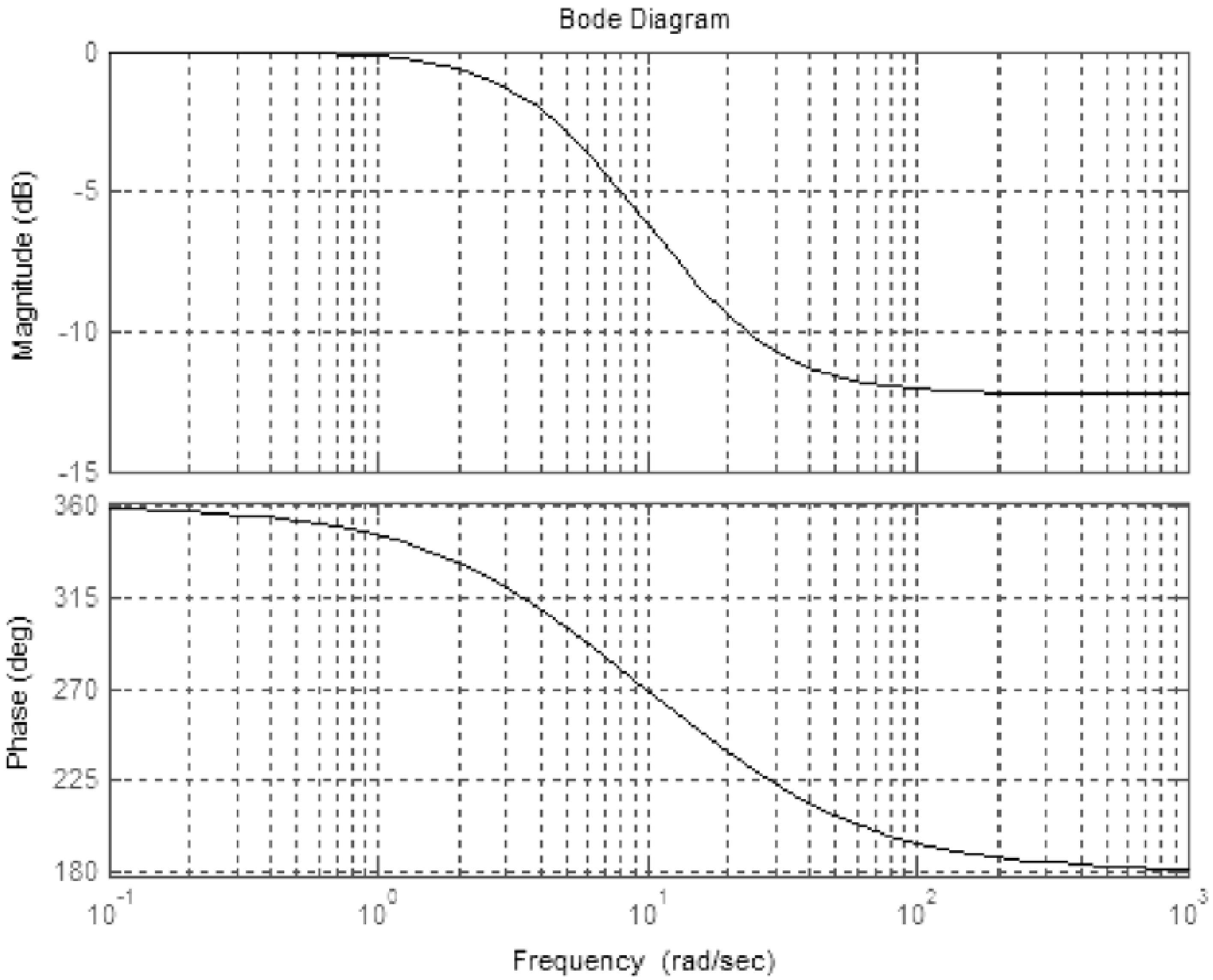

Figure 8.12 Bode diagram of the system

Bode diagrams

Example 8.14

The loop transfer function of a unity feedback sampled data system is given by

$$G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}.$$

Draw the Bode diagram and analyse the stability of the system. Assume that $T = 1$ s.

Solution

Using the transformation

$$z = \frac{1 + (T/2)w}{1 - (T/2)w} = \frac{1 + 0.5w}{1 - 0.5w},$$

we get

$$G(w) = \frac{0.368(1 + 0.5w/1 - 0.5w) + 0.264}{(1 + 0.5w/1 - 0.5w)^2 - 1.368(1 + 0.5w/1 - 0.5w) + 0.368}$$

or

$$G(w) = -\frac{0.0381(w - 2)(w + 12.14)}{w(w + 0.924)}.$$

Bode diagrams

To obtain the frequency response, we can replace w with jv, giving

$$G(jv) = -\frac{0.0381(jv-2)(jv+12.14)}{jv(jv+0.924)}.$$

The magnitude is then

$$|G(jv)| = \frac{0.0381\sqrt{v^2+2^2}\sqrt{v^2+12.14^2}}{v\sqrt{v^2+0.924^2}}$$

and

$$\angle G(jv) = \tan^{-1}\frac{v}{2} + \tan^{-1}\frac{v}{12.14} - 90 - \tan^{-1}\frac{v}{0.924}.$$

Bode diagrams

The Bode diagram is shown in Figure 8.13. The system is stable with a gain margin of 5 dB and a phase margin of 26°.

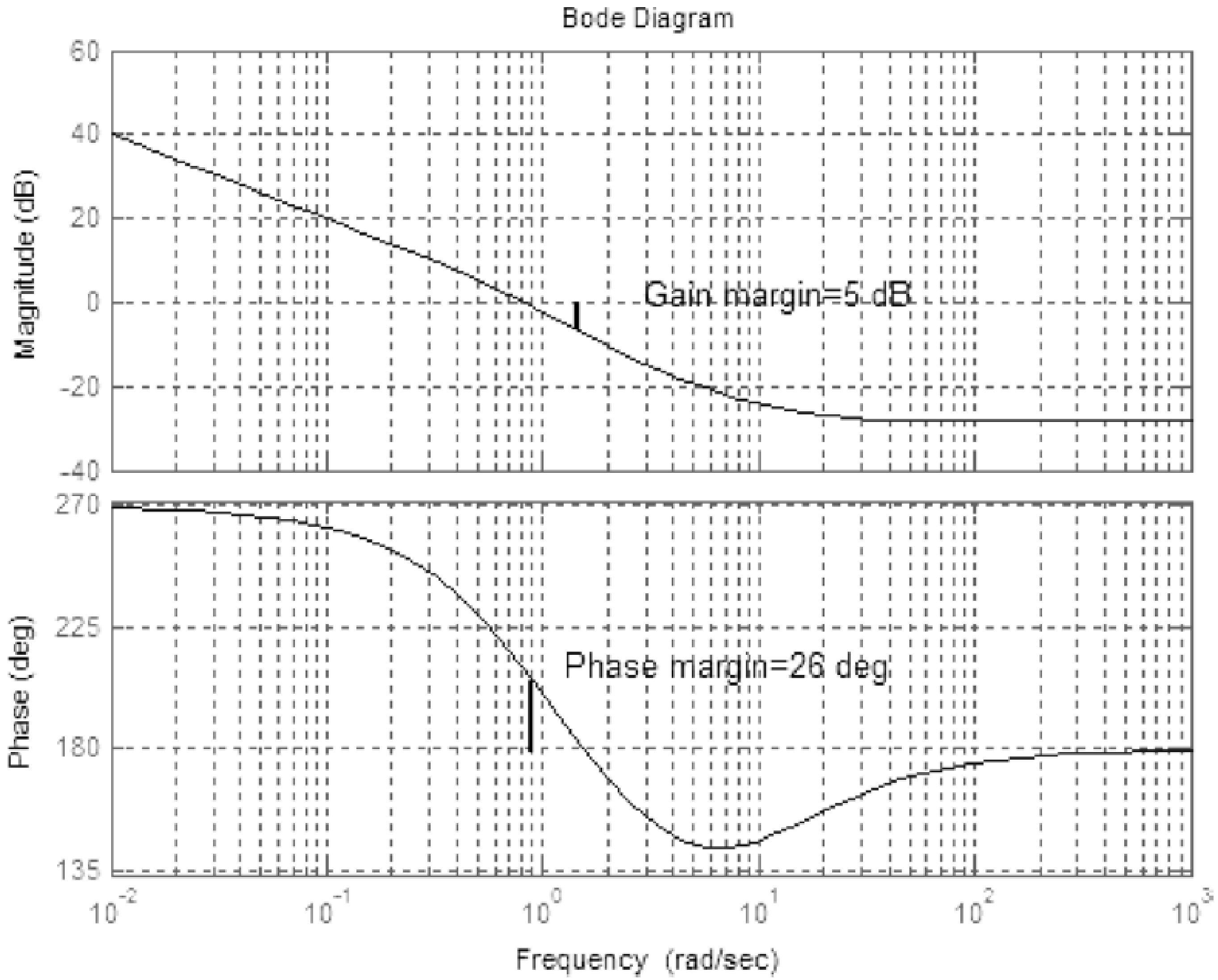

Figure 8.13 Bode diagram of the system